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Fiber Shape Theory

by

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The theory of fibrations has been studied by many mathematicians and many concepts of fibrations have been important tools in the study of geometric topology and algebraic topology. After the works of cell-like maps and UV^n-maps by R. C. Lacher [17], S. Armentrout and T. M. Price [1], and G. Kozlowski [15] and [16], recently, D. Coram and P. Duvall ([5] and [6]) introduced the notion of approximate fibration, which is the shape theoretic analogue of fibration. Later, S. Mardešić and T. B. Rushing ([18] and [19]) further generalized this concept with the introduction of shape fibration.

We shall introduce new categories "Fiber shape categories", which are shape theoretic categories analogous to the fiber homotopy category, and we study geometric properties of cell-like maps, approximate fibrations and shape fibrations. Throughout this paper, all spaces are metrizable and all maps are continuous. By an ANR (resp. AR), we denote an absolute neighborhood retract (resp. absolute retract) for the class of metrizable spaces. By I we mean the unit intervall [0,1] and by Q the Hilbert cube.

Let f and g be maps from a space X to a compactum (Y,d). The *sup-metric* d is given by $d(f,g)=\sup\{d(f(x),g(x))\mid x\in X\}$. Let E, E' and B be compacta contained in AR's X, X' and Y, respectively. Suppose that $\widetilde{p}\colon X\to Y$ and $\widetilde{p}'\colon X'\to Y$ are extensions of maps $p\colon E\to B$ and $p'\colon E'\to B$, respectively. A fundamental sequence [1] $\underline{f}=\{f_n,E,E'\}_{X,X'}$ is a *fiber fundamental sequence over* B [10] if for any $\varepsilon>0$ and any neighborhood U' of E' in X' there is a neighborhood U of E in X and a positive integer n_0 such that for each $n\geqq n_0$ there is a homotopy $F\colon U\times I\to U'$ such that $F(x,0)=f_{n_0}(x)$, $F(x,1)=f_n(x)$ for $x\in U$ and $d(\widetilde{p}'F(x,t),\widetilde{p}(x))<\varepsilon$ for $x\in U$, $t\in I$. Two fiber fundamental sequences $\underline{f}=\{f_n,E,E'\}_{X,X'}$ and $\underline{g}=\{g_n,E,E'\}_{X,X'}$ are *fiber homotopic* if for any $\varepsilon>0$ and any neighborhood U' of E' in X' there is a neighboehood U of E in X and a positive integer n_0 such that for each $n\geqq n_0$ there is a homotopy $K\colon U\times I\to U'$ such that $K(x,0)=f_n(x)$, $K(x,1)=g_n(x)$ for $x\in U$ and $d(\widetilde{p}'K(x,t),\widetilde{p}(x))<\varepsilon$ for $x\in U$, $t\in I$. A map $p\colon E\to B$ is *fiber shape equivalent* to a map $p'\colon E'\to B$ if there are fiber fundamental sequences $\underline{f}=\{f_n,E,E'\}_{X,X'}$ and $\underline{h}=\{h_n,E',E\}_{X',X}$ such that $\underline{gf}$ is fiber homotopic to $\underline{1}_E$ and $\underline{fg}$ is fiber homotopic to $\underline{1}_{E'}$, where $\underline{1}_E$ and $\underline{1}_{E'}$ denote the fiber fundamental sequences induced by the identities $1_E\colon E\to E$ and $1_{E'}\colon E'\to E'$, respectively. Such $\underline{f}$ is called a *fiber shape equivalence*. A map $p\colon E\to B$ is *shape shrinkable* if p induces a fiber shape equivalence from p to the identity $1_B\colon B\to B$. We will prove that $p\colon E\to B$ is shape shrinkable iff p is a hereditary shape equivalence. We denote by M_B the category whose objects are all maps of compacta to the fixed compactum B and whose morphisms are fiber homotopy classes of fiber fundamental sequences over B.

THEOREM 1. Let $p\colon E\to B$ and $p'\colon E'\to B$ be approximate fibrations between compact ANRs. If a fiber fundamental sequence $\underline{f}$ from p to p' is a shape equivalence [2], then it is a fiber shape equivalence. In particular, if a fiber map $f\colon E\to E'$ is a homotopy equivalence, then it is a fiber shape equivalence.

COROLLARY 2. Let $p\colon E\to B$ be a map between compact ANRs. Then the following are equivalent.

(1) p is a cell-like map.
(2) p is a homotopy equivalence and an approximate fibration.
(3) p is shape shrinkable.
(4) p is a hereditary shape equivalence.

By the results of G. Kozlowski [16] and Corollary 2, we have the following.

COROLLARY 3. Let $p\colon E\to B$ be a map between compacta. Then p is shape shrinkable iff p is a hereditary shape equivalence.

In 1972, T. A. Chapman (Fund. Math., 76 (1972), 181-193) proved the Complement Theorem, i.e., if X and Y are Z-sets in the Hilbert cube Q, then X and Y have the same shape iff Q-X and Q-Y are homeomorphic. Now, we give a much sharper form of the Complement Theorem in the category M_B.

THEOREM 4. Let E, E' and B be compacta and let $E, E' \subset Q$ be Z-sets. Then a map $p: E \to B$ is fiber shape equivalent to a map $p': E' \to B$ iff there is a homeomorphism $h: Q-E \cong Q-E'$ such that for each $b \in B$ and each neighborhood W' of $p'^{-1}(b)$ in Q, there is a neighborhood W of E in Q such that $h(W-E) \subset W'-E'$.

COROLLARY 5. Let E and B be Z-sets in Q. Then a map $p: E \to B$ is shape shrinkable iff there is an extension $\tilde{p}: Q \to Q$ of p such that $\tilde{p}|Q-E: Q-E \cong Q-B$ is a homeomorphism.

REMARK 6. If B is the one point set, the category M_B is the same as shape category [2], [3].

Next, we shall define a category FR_B as follows. For a subset E of a space X, E is <u>unstable</u> in X if there is a homotopy $H: X \times I \to X$ such that $H(x,0) = x$ and $H(x,t) \in X-E$ for $x \in X$, $0 < t \leq 1$. Let $p: E \to B$ and $p': E' \to B$ be maps between compacta and let E and E' be subsets of compacta X and X', respectively. A map $f: X-E \to X'-E'$ is an $F(p,p')$-<u>map</u> [11] if for each $b \in B$ and each neighborhood W' of $p'^{-1}(b)$ in X' there is a neighborhood W of $p^{-1}(b)$ in X such that $f(W-E) \subset W'-E'$. Two $F(p,p')$-maps $f,g\colon X-E \to X'-E'$ are $F(p,p')$-homotopic ($f \underset{F(p,p')}{\simeq} g$) if there is a homotopy $H\colon (X-E)\times I \to X'-E'$ such that $H(x,0)=f(x)$, $H(x,1)=g(x)$ for $x \in X-E$ and each neighborhood W' of $p'^{-1}(b)$ in X' there is a neighborhood W of $p^{-1}(b)$ in X such that $H((W-E)\times I) \subset W'-E'$. Such a homotopy H is called an $F(p,p')$-homotopy. Now we need the following lemma (see [14]).

LEMMA 7. Let X and X' be compact ARs containing E as an unstable closed subset, respectively. Then there is a map $\varphi(X,X')\colon X \to X'$ such that

$$(*) \qquad \varphi(X,X')|E = 1_E \quad \text{and} \quad \varphi(X,X')(X-E) \subset X'-E'.$$

If φ_1, $\varphi_2\colon X \to X'$ satisfy the condition $(*)$, then there is a homotopy $H\colon X\times I \to X'$ such that $H(x,0)=\varphi_1(x)$, $H(x,1)=\varphi_2(x)$ for $x \in X$ and $H(x,t)=x$ for $x \in E$, $t \in I$ and $H((X-E)\times I) \subset X'-E'$. In particular, for any map $p\colon E \to B$ between compacta $\varphi(X,X')|X-E\colon X-E \to X'-E'$ is an $F(p,p)$-map and $H|(X-E)\times I\colon (X-E)\times I \to X'-E'$ is an $F(p,p)$-homotopy.

For a compactum E, we denote by $m(E)$ the set of compact ARs containing E as an unstable subset. Let $p\colon E \to B$ and $p'\colon E' \to B$ be maps between compacta and let X_1, $X_2 \in m(E)$ and X_1', $X_2' \in m(E')$. An $F(p,p')$-map $f\colon X_1-E \to X_1'-E'$ is $F(p,p')$-equivalent to an $F(p,p')$-map $g\colon X_2-E \to X_2'-E'$ if

$$\varphi(X_1',X_2')\,|\,(X_1'-E')\circ f \;\underset{F(p,p')}{\widetilde{\qquad}}\; g\circ\varphi(X_1,X_2)\,|\,(X_1-E).$$

Objects of FR_B are all maps of compacta to B and for objects $p\colon E\to B$ and $p'\colon E'\to B$ of FR_B, morphisms from p to p' in FR_B are $F(p,p')$-equivalence classes of collections of $F(p,p')$-maps $f\colon X-E\to X'-E'$, $X\in m(E)$, $X'\in m(E')$. Then FR_B forms a category (see [11]).

REMARK 8. Note that if B is the one point set, the category FR_B is the same as strong (or fine) shape category ([7], [8] and [13, 14]).

THEOREM 9. Let $p\colon E\to B$ and $p'\colon E'\to B$ be maps between compacta. Then the following are equivalent.

(1) p is isomorphic to p' in M_B.

(2) p is isomorphic to p' in FR_B.

THEOREM 10. A map $p\colon E\to B$ between compacta is shape shrinkable iff p is an isomorphism from p to 1_B in FR_B.

THEOREM 11. Let $p\colon E\to B$ and $p'\colon E'\to B$ be maps between compacta and let $f\colon p\to p'$ be a morphism in FR_B. If B has a finite closed cover $\{B_i\}_{i=1,2,,n}$ such that for each i the restriction $f|p^{-1}(B_i)\colon p|p^{-1}(B_i)\to p'|p'^{-1}(B_i)$ is an isomorphism in FR_{B_i}, then $f\colon p\to p'$ is an isomorphism in FR_B.

Note that if every $F(p,p')$-map and $F(p,p')$-homotopy are proper map and proper homotopy, respectively. Hence, there is a forgetful functor $T\colon FR_B\to$ s-Sh such that $T(p)=E$

for each object $p\colon E \to B$ of FR_B. Then we have the following.

THEOREM 12. Let $p\colon E \to B$ and $p'\colon E' \to B$ be shape fibrations between compacta. Then a morphism $f\colon p \to p'$ in FR_B is an isomorphism iff $T(f)\colon E \to E'$ is a strong shape equivalence.

As a special case of Theorem 12, we have the next corollary. The corollary is proved by F. Cathey, independently.

COROLLARY 13. Let $p\colon E \to B$ be a map between compacta. Then the following are equivalent.

(1) p is a shape fibration and a strong shape equivalence.
(2) p is shape shrinkable (a hereditary shape equivalence).

By using Theorem 11 and 12, we have the following.

THEOREM 14. Let $p\colon E \to B$ and $p'\colon E' \to B$ be shape fibrations between compacta. Suppose that B is a continuum with a finite closed cover consisting of FARs or B is a connected ANR. Then a morphism $f\colon p \to p'$ of FR_B is an isomorphism in FR_B iff for some $b \in B$, the restriction $T(f|p^{-1}(b))\colon p^{-1}(b) \to p'^{-1}(b)$ of $T(f)$ to $p^{-1}(b)$ is an isomorphism in s-Sh.

COROLLARY 15. Let $p\colon E \to B$ be a map between compacta. Suppose that B is an ANR or B has a finite closed cover consisting of FARs. Then the following are equivalent.

(1) p is a cell-like map and a shape fibration.

(2) p is shape shrinkable.

REMARK 16. In the statement of Corollary 15, we cannot replace the condition about B by "pointed movable" (see [8]).

THEOREM 17. Let E, E' and B be compacta and dim $B<\infty$. Suppose that $p\colon E\to B$ and $p'\colon E'\to B$ are strongly regular mappings with ANR fibers (see [9] for the definition of strongly regular mapping). Then p is fiber homotopy equivalent to p' iff p is isomorphic to p' in FR_B. Moreover, if a fiber map $f\colon E\to E'$ from p to p' induces an isomorphism in FR_B, then it is a fiber homotopy equivalence.

THEOREM 18. Let E and B be compacta and dim $B<\infty$. If $p\colon E\to B$ is a strongly regular mapping with ANR fibers, then p is a shape fibration.

REMARK 19. In the statement of Theorem 18, we cannot omit the condition "dim $B<\infty$".

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